An error covariance model for multivariate data assimilation in tidal regimes

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4th Coastal Altimetry Workshop, Porto, Oct 15, 2010

Motivation

1. Near-coastal data assimilation is difficult because coastal regions are rarely covered by observations with space-time resolution adequate for resolving processes on scales of several hours/km: Regional OGCMs have ***large background errors near the coast*** .
2. The background error fields may have ***significant correlations at scales comparable with the size of the domain***, especially in tidally-dominated regions (estuaries, fiords).
3. ***Satellite data*** combined with ***gliders/radars*** may provide coverage of the coastal domains which is dense enough to retrieve the principal background error covariance structures.

OBJECTIVES

1. Develop an error covariance model with a capability of handling large background errors correlated at scales comparable with the size of the domain
2. Design a ***hybrid*** adaptive algorithm which detects robust structures in the error fields and provides automatic estimation of the background error variance.

Outline

1. Hybrid background error covariance (BEC) modeling

2. The BEC model description
   a. Separation of the dynamical and heuristic BECs
   b. Semi-implicit scheme for the heuristic covariance operator
   c. Assessing the rank of the dynamical part of the covariance
   d. Weighting dynamical and heuristic BECs against observation errors

3. Experiments with simulated data

4. Experiments with real data

5. Summary

Hybrid covariance modelling (0)

$$J = \frac{1}{2}\left[\delta \mathsf{x}^T \mathbf{B}^{-1} \delta \mathsf{x} + (\boldsymbol{H}\delta \mathsf{x} - \delta d)^T \boldsymbol{R}^{-1} (\boldsymbol{H}\delta \mathsf{x} - \delta d)\right] \rightarrow \min_{\delta \mathsf{x}}$$

$$\delta \mathsf{x} = \mathsf{x} - \mathsf{x}_b \qquad \delta d = d - \boldsymbol{H}\mathsf{x}_b$$

Woodbury identity

$$\delta \mathsf{x} = \left[\mathbf{B}^{-1} + \boldsymbol{H}^T \boldsymbol{R}^{-1} \boldsymbol{H}\right]^{-1} \boldsymbol{H}^T \boldsymbol{R}^{-1} \delta d \quad \longleftrightarrow \quad \delta \mathsf{x} = \mathbf{B}\boldsymbol{H}^T \left[\boldsymbol{H}\mathbf{B}\boldsymbol{H}^T + \boldsymbol{R}\right]^{-1} \delta d$$

Model space formulation | **Obs space formulation**

0. The structure of B is poorly known: only a small number of eigenvectors can be captured with confidence.
1. Formulations are equivalent only if the increment δx is restricted to the subspace spanned by the eigenvectors of **B**.
2. Result of assimilation crucially depends on the model of **B** (or $\mathbf{B}^{-1}$)

Hybrid covariance modelling (1)

$$J(\delta \mathsf{x}) = \frac{1}{2}\left[\delta \mathsf{x}^\dagger \mathsf{B}^{-1}\delta \mathsf{x} + (H\delta \mathsf{x} - \delta y)^\dagger R^{-1}(H\delta \mathsf{x} - \delta y)\right] \rightarrow \min_{\delta \mathsf{x}}$$

$$\delta y = H\mathsf{x}_b - d$$

$$\mathsf{B}^{-1} = \alpha \mathsf{B}_m^{-1} + \beta \mathsf{B}_0^{-1}$$

$$\mathsf{B}^{-1} = \alpha P \Lambda_m^{-1} P^\dagger + \beta P_\perp \mathsf{B}_0^{-1} P_\perp{}^\dagger$$

$$P_\perp = \mathsf{I} - PP^\dagger$$

$$\mathsf{B} = \alpha \mathsf{B}_m + \beta \mathsf{B}_0$$

$$\mathsf{B}_0 = \exp(\nabla \nu \nabla t)$$

heuristic (Gaussian)

$$\mathsf{B}_m = P \Lambda_m P^\dagger$$

dynamical (derived from model statistics)

$$\mathsf{B} = \frac{1}{\alpha} P \Lambda_m P^\dagger + \frac{1}{\beta}[P_\perp \exp(-\mathsf{D}t) P_\perp^\dagger]^{-1}$$

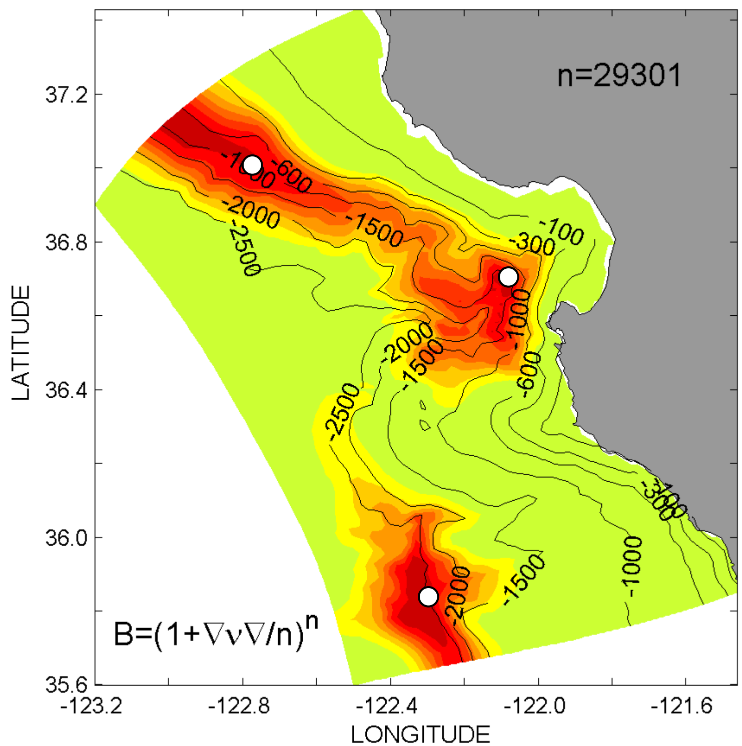
$$\mathsf{D} = \nabla^\dagger \nu \nabla$$

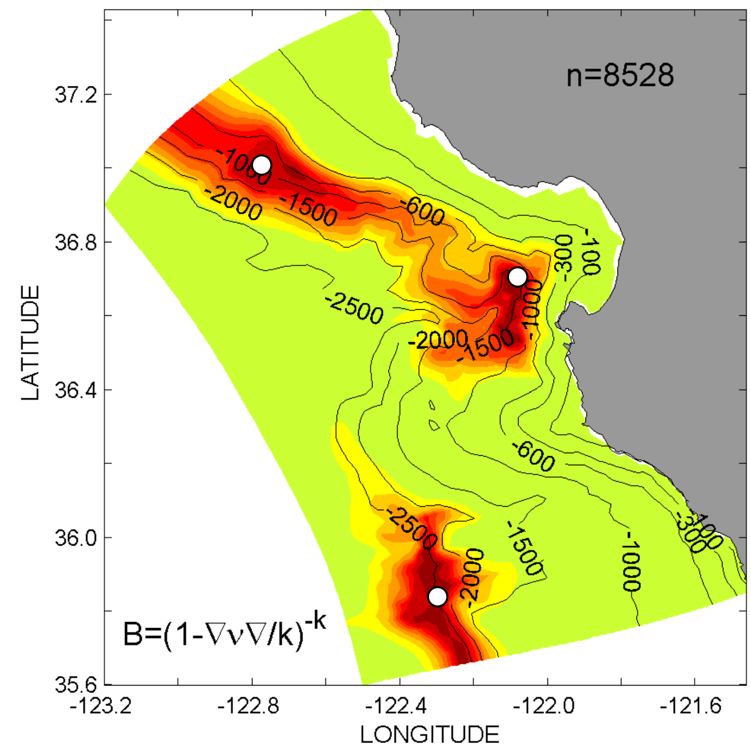
Computation of the Gaussian covariances B_0

Explicit vs implicit “time integration”:

$\exp(\mathbf{D}t) \sim (\mathbf{I} + \mathbf{D}t/n)^n$

$\exp(\mathbf{D}t) \sim (\mathbf{I} - \mathbf{D}t/m)^{-m}$





$\max[\rho/\delta x] = 11$ $\quad m=2 \quad$ $CPU_{obs} / CPU_{mod} \sim 20m\,[\rho/\delta x]^{1-m} \sim 4$

Estimation of α (magnitude of B_m)

$$\boxed{\delta x = P\delta e}$$

$$\boxed{J = \frac{1}{2}\left[\delta e^\dagger P^\dagger B^{-1} P\delta e + (HP\delta e - \delta y)^\dagger (HP\delta e - \delta y)\right] \to \min_{\delta e \in \mathcal{R}^m}}$$

$$\left[P^\dagger B^{-1} P + Q\right]\delta e = E\delta y$$

$$\left[\alpha\Lambda_m^{-1} + Q\right]\delta e = E\delta y$$

$$\boxed{\begin{array}{ll} E \equiv P^\dagger H^\dagger & P_\perp P = 0 \\ Q = EE^\dagger & P^\dagger P = I_m \end{array}}$$

$$\boxed{\langle \delta e \delta e^\dagger \rangle = (\alpha\Lambda_m^{-1} + Q)^{-1} E Y E^\dagger (\alpha\Lambda_m^{-1} + Q)^{-1}}$$

$$\boxed{\begin{array}{l} Y = \langle \delta y \delta y^\dagger \rangle \\ Y = HBH^\dagger + I_K \end{array}}$$

$$EYE^\dagger = EHBH^\dagger E^\dagger + EE^\dagger = \frac{1}{\alpha} Q\Lambda_m Q^\dagger + \cancel{\frac{1}{\beta} EH[P_\perp B_0^{-1} P_\perp]^{-1} H^\dagger E^\dagger} + Q.$$

$$EYE^\dagger = \frac{1}{\alpha} Q\Lambda_m Q^\dagger + Q$$

$$P^\dagger H^\dagger H [P_\perp B_0^{-1} P_\perp]^{-1} H^\dagger H P \simeq 0$$

$$\boxed{\langle \delta e \delta e^\dagger \rangle = \left[\alpha\Lambda_m^{-1} + Q\right]^{-1} \left[\frac{1}{\alpha} Q\Lambda_m Q^\dagger + Q\right] \left[\alpha\Lambda_m^{-1} + Q\right]^{-1}}$$

$$\boxed{\begin{array}{l} \langle \delta e \delta e^\dagger \rangle \simeq \dfrac{1}{\alpha}\Lambda_m \\ |Q| \gg \alpha / |\Lambda_m| \end{array}}$$

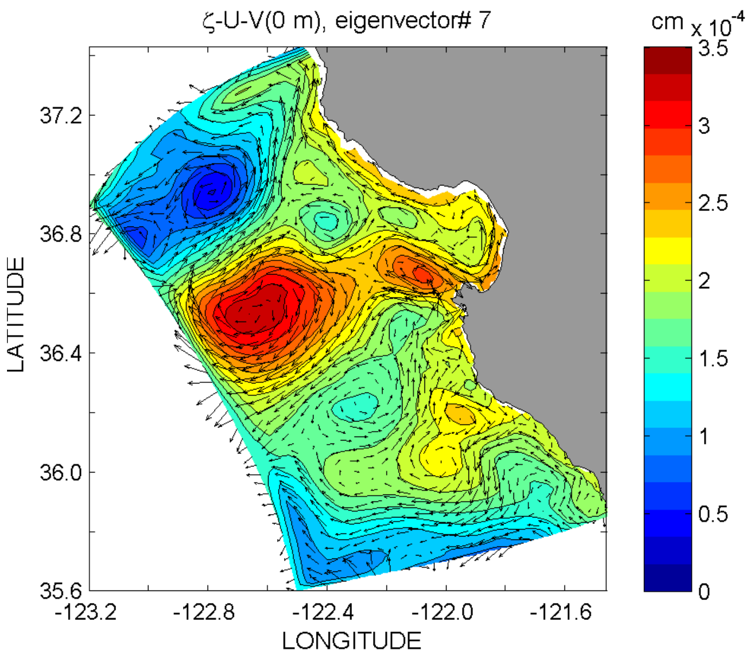
Estimation of the number of modes *m*

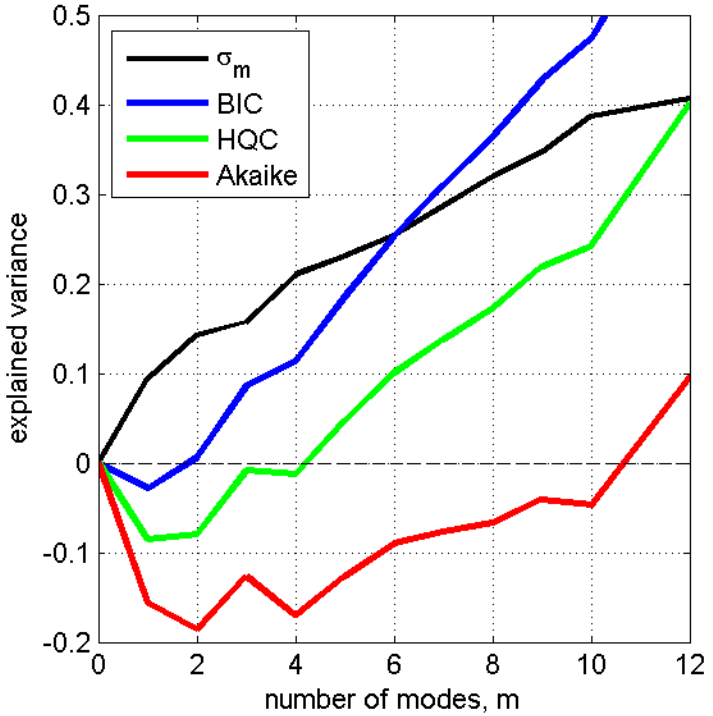
$$\mathcal{C}_{\mathrm{BIC}}(m) = m + \frac{N}{\ln N}\ln\sigma_m^2$$

$$\mathcal{C}_{\mathrm{Akaike}}(m) = m + \frac{N}{2}\ln\sigma_m^2$$

$$\mathcal{C}_{\mathrm{HQC}}(m) = m + \frac{N}{2\ln\ln N}\ln\sigma_m^2$$

σ_m^2 - *part of δy variance described by* ***m*** *modes (average over N samples)*





Estimation of β (magnitude of B_0)

$$Y = \langle \delta y \delta y^\dagger \rangle = H\mathrm{B}H^\dagger + \mathrm{I}_K$$

$$\boxed{\beta: \quad \mathrm{Tr}\, Y = \mathrm{Tr}\,(H\mathrm{B}H + \mathrm{I}_K)}$$

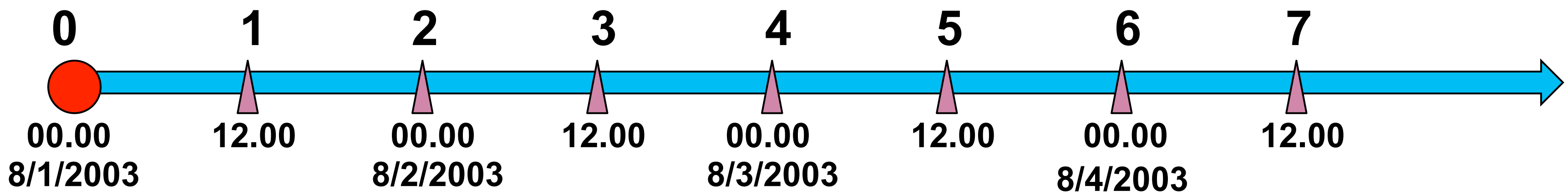
$$\mathrm{Tr}\langle \delta y \delta y^\dagger \rangle = \mathrm{Tr}\left[\frac{1}{\alpha} H P \Lambda_m P^\dagger H^\dagger + \frac{1}{\beta} H [P_\perp \exp(-\mathrm{D}t) P_\perp{}^\dagger]^{-1} H^\dagger\right] + K$$

$$\boxed{\beta = \frac{\mathrm{Tr}\{H[P_\perp \exp(-\mathrm{D}t) P_\perp{}^\dagger]^{-1} H^\dagger\}}{\langle \delta y \delta y^\dagger \rangle - K - \mathrm{Tr}[E^\dagger \Lambda_m E]/\alpha}}$$

ANALYSIS EQUATION

$$\boxed{\left[\alpha P \Lambda_m^{-1} P^\dagger + \beta P_\perp \exp(-\mathrm{D}t) P_\perp^\dagger + H^\dagger H\right] \delta \mathrm{x} = H^\dagger \tilde{d}}$$

Assimilation scheme (1)

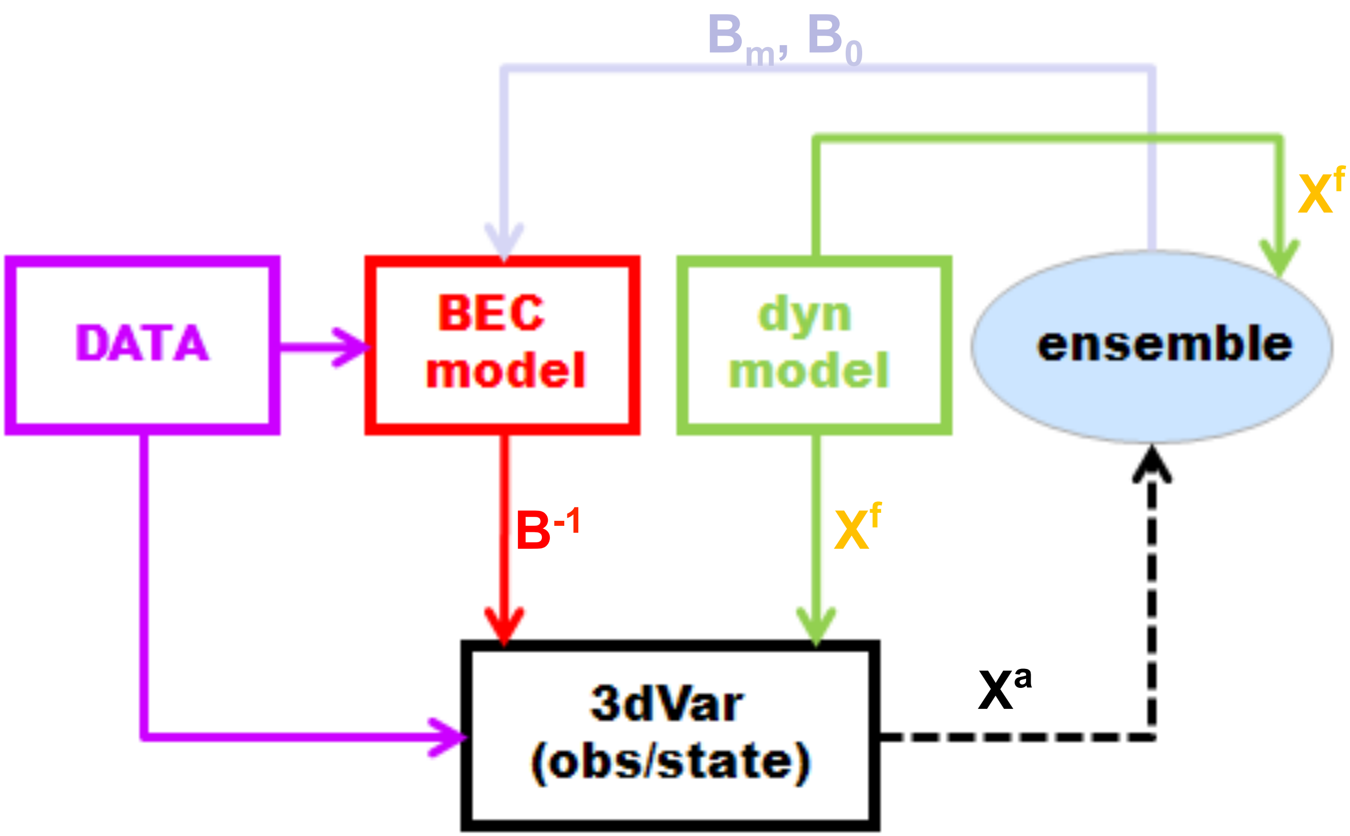


1. $\{X^{a,f}_{1\ldots n-1}\}, B_{n-1} \rightarrow X^{a}_{n}, X^{f}_{n}$ perform analysis + forecast at $t=t_n$

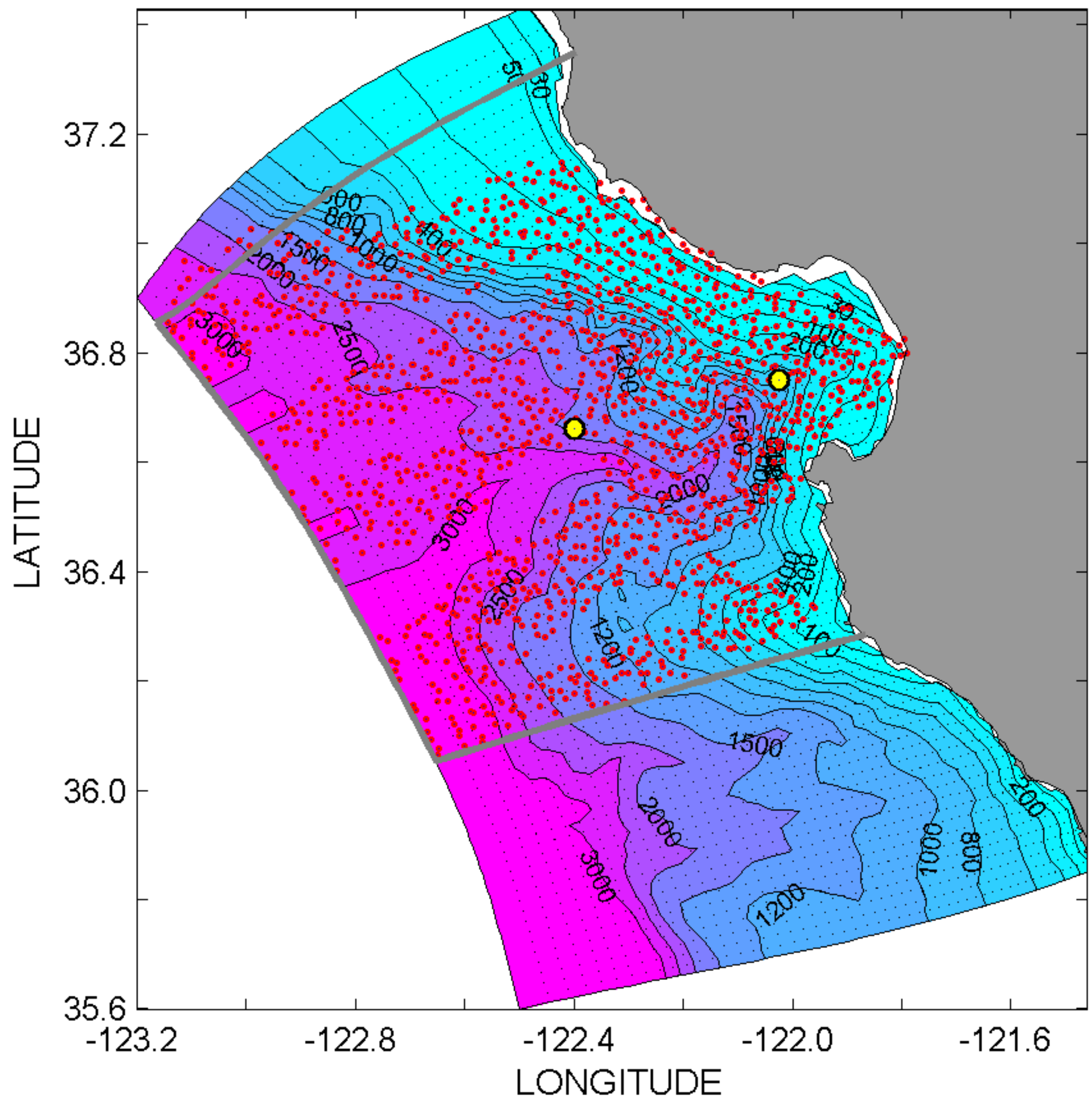
2. $\{X^{f}_{1\ldots n-1}\}, B_{n-1} \rightarrow \{X^{f}_{1\ldots n}\}, B_n$ update the ensemble/covariance

3. $\{X^{a,f}_{1\ldots n}\}, B_n \rightarrow X^{a}_{n+1}, X^{f}_{n+1}$ perform analysis + forecast at $t=t_{n+1}$

Assimilation scheme (2)



Twin data experiments (1)

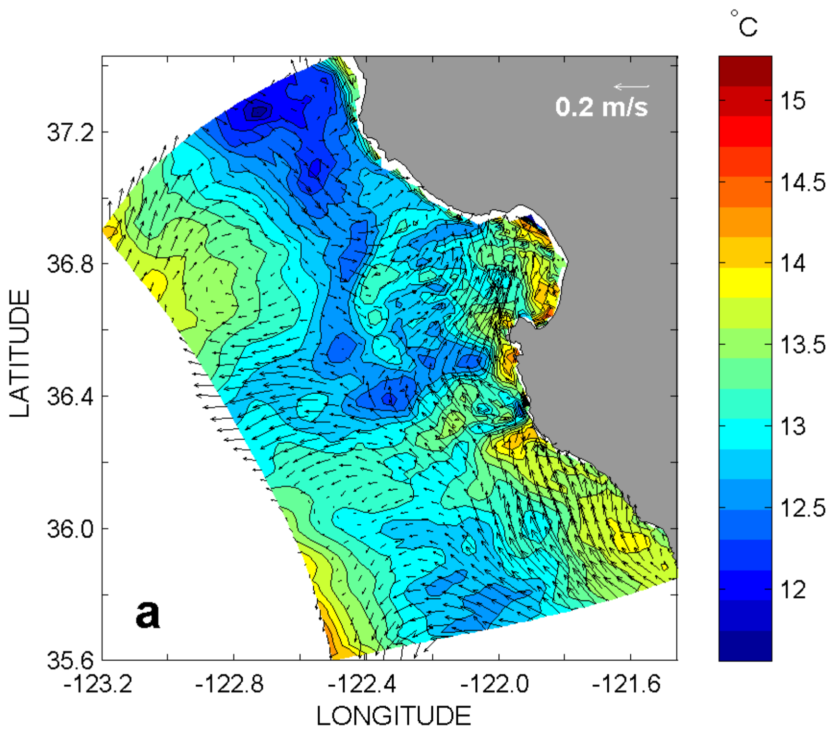


Gliders:
duration: 27 days
K ~1.5 x 10^5

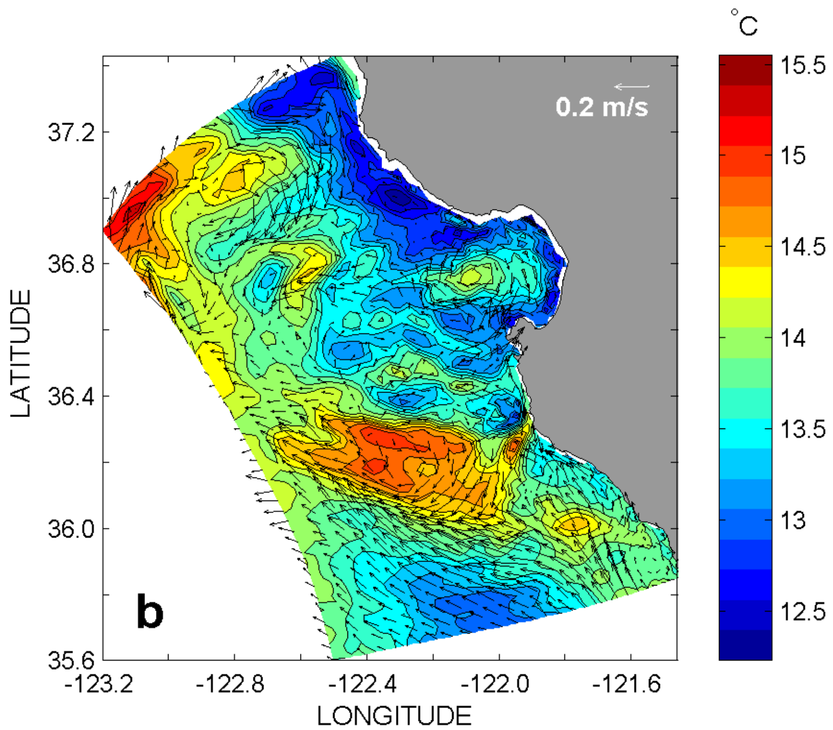
Model:
NCOM 1.5km 40 levels
M = 515,450

Validation:
a) 12hr forecast errors
b) Two moorings with
18 levels U/V
11 levels T/S

Twin data experiments (2)



“True”



First guess

Skill assessment

Metric

$$G = \mathrm{diag}\{g_T, g_S, g_u, g_v\} \qquad g_\xi(\mathbf{x}) = \overline{\left[\xi(\mathbf{x}) - \overline{\xi(\mathbf{x})}\right]^2}^{1/2}$$

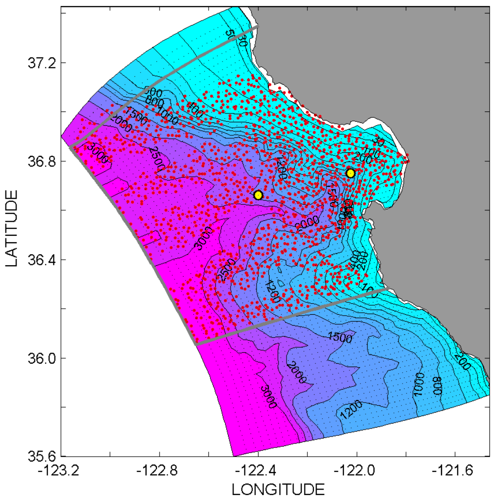
Distances

$$r_\xi^s(\mathsf{x}_1, \mathsf{x}_2) = \langle(\xi_1 - \xi_2)^2 g_\xi^{-2}\rangle_S^{1/2} \qquad r_\xi^g(\mathsf{x}_1, \mathsf{x}_2) = \langle(\xi_1 - \xi_2)^2 R^{-1}\rangle_g^{1/2}$$

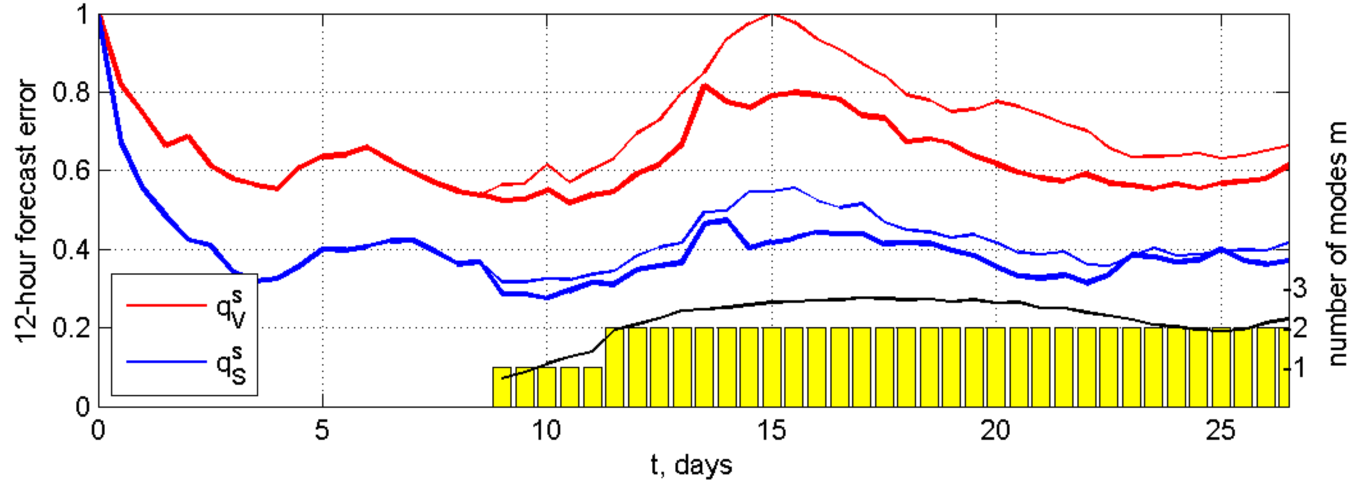
state space   obs space

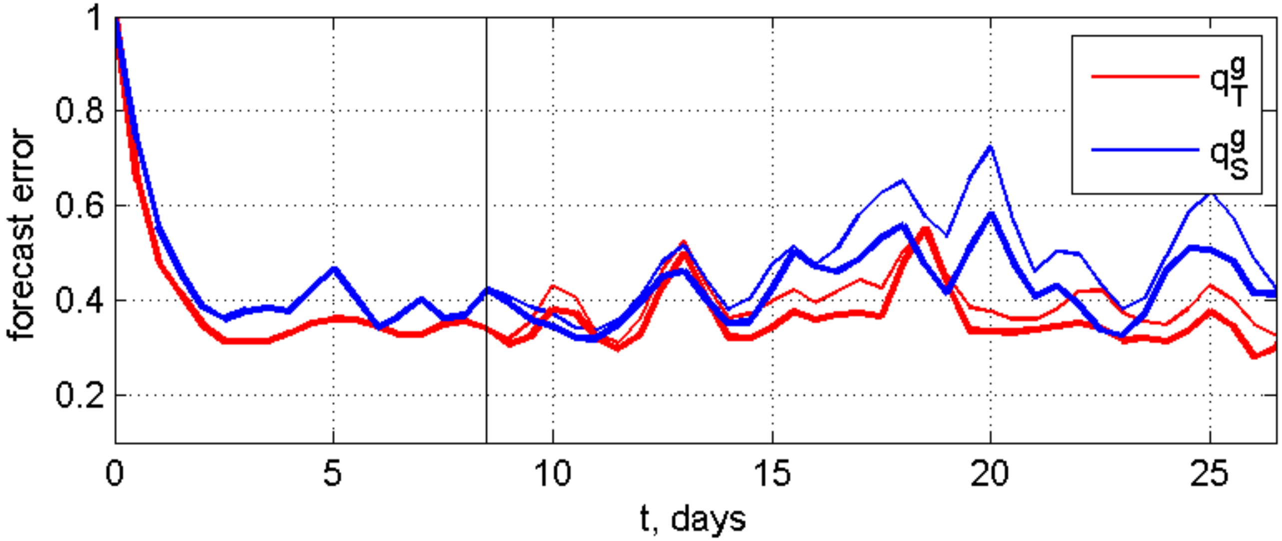
Skill

$$q_\xi^{g,s}(t) = \frac{r_\xi^{g,s}(\mathsf{x}^t, \mathsf{x}^f)|_t}{r_\xi^{g,s}(\mathsf{x}^t, \mathsf{x}^{fg})|_0}$$

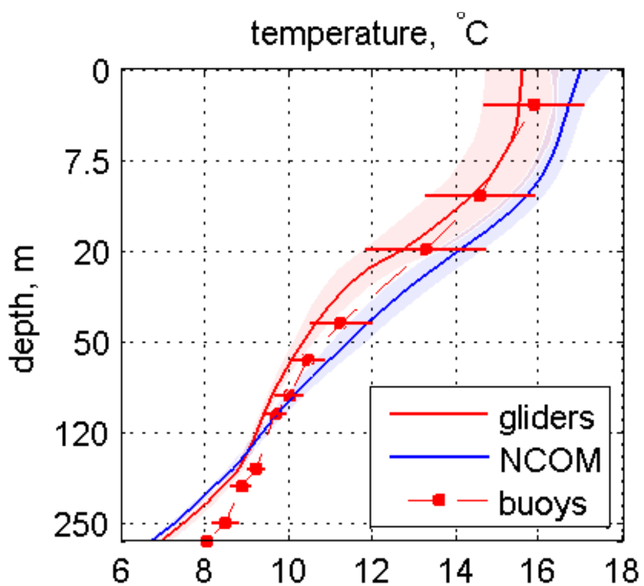


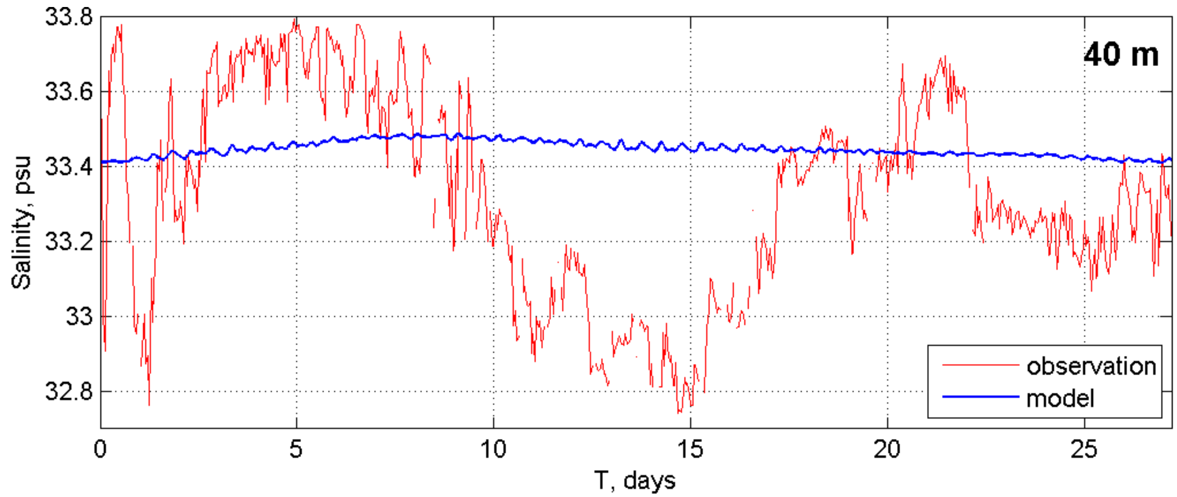
Twin data experiments (3)

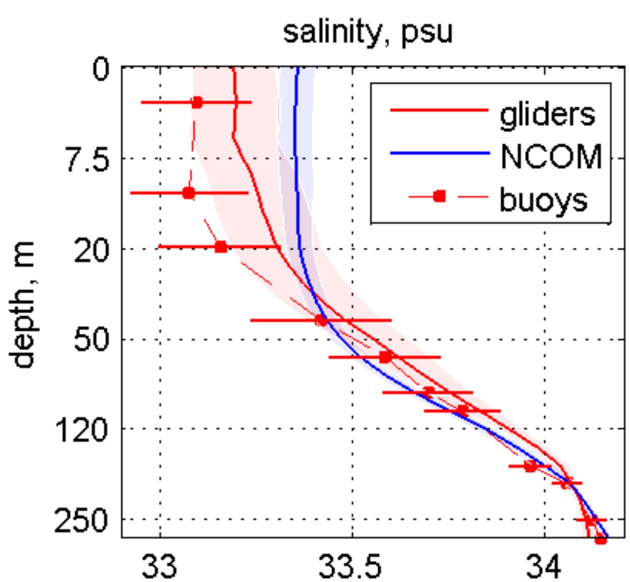




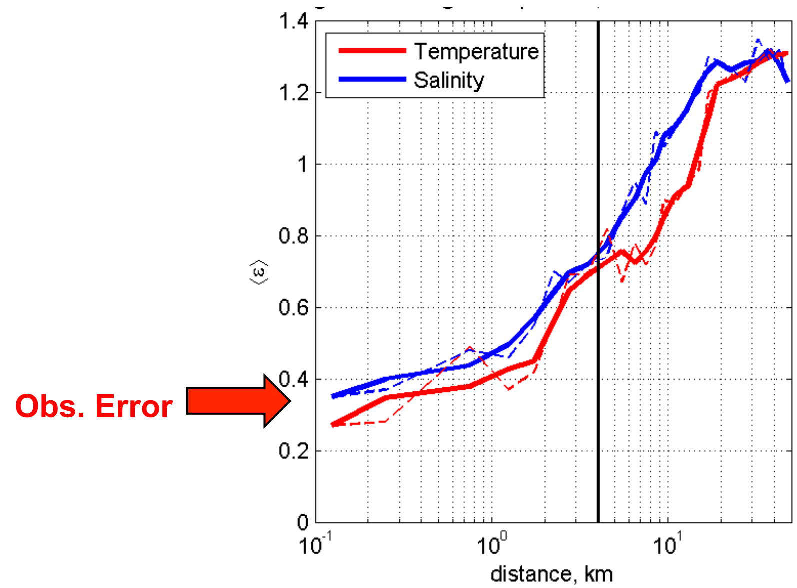
Real data experiments (1)



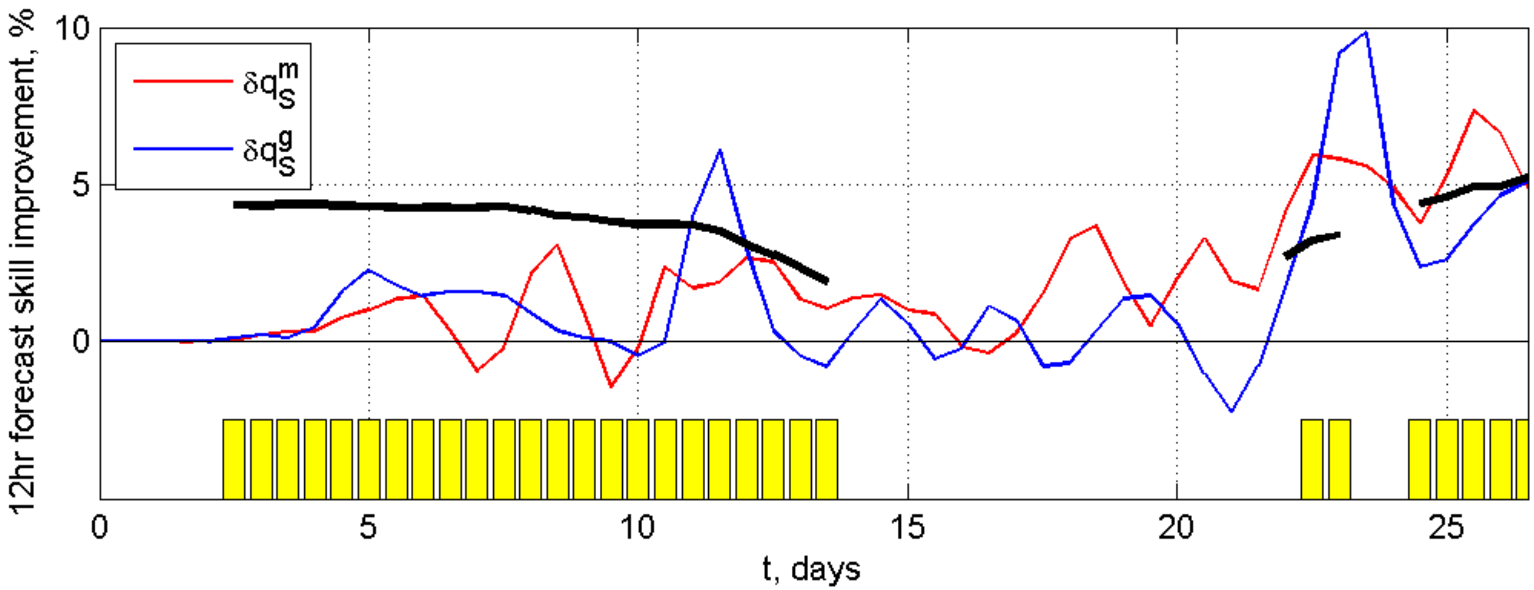


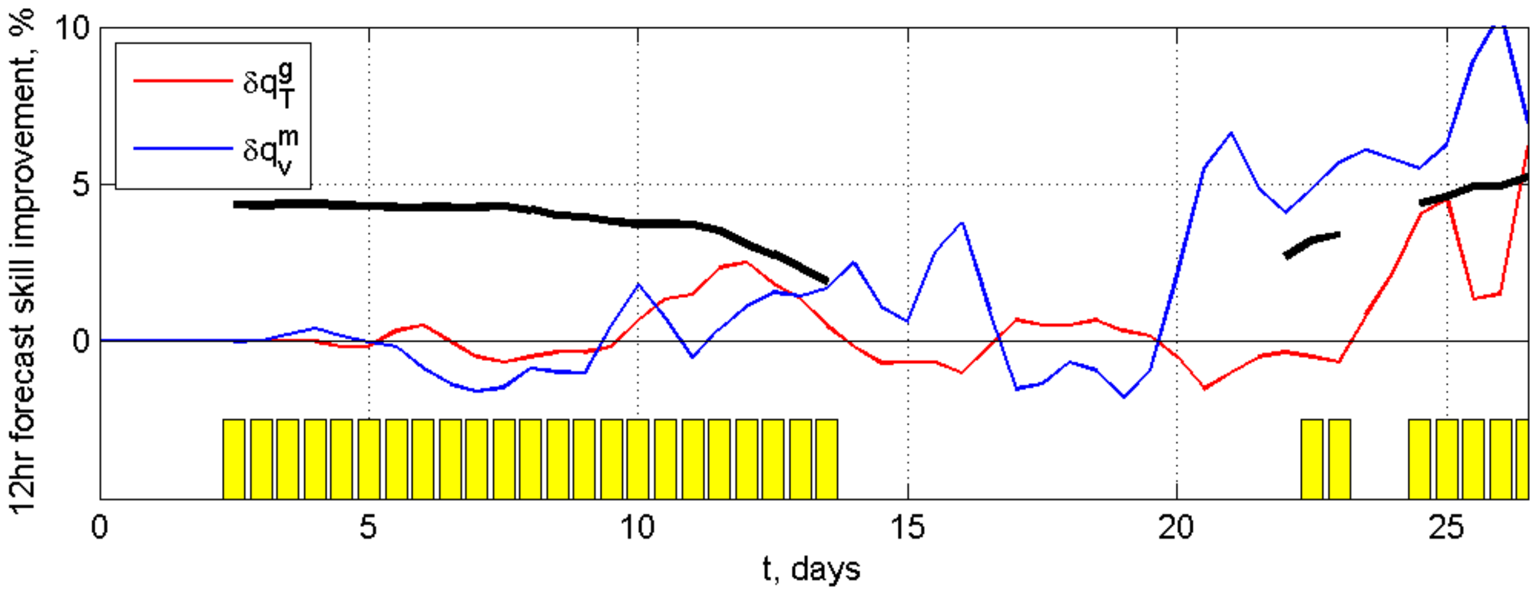


Glider-mooring discrepancies



Real data experiments (2)





Summary

1. **A hybrid background error covariance model has been proposed and tested with simulated and real data in the Monterrey Bay**

2. **The distinctive features are:**
   a) formulation in terms of the inverse error covariances
   b) adaptive determination of the rank($\mathbf{B}_m$) via information criterion
   c) restriction of $\mathbf{B}_0$ to the null space of $\mathbf{B}_m$
   d) adaptive definition of the weighting factors via separate analyses of the innovation vector statistics in the null space of $\mathbf{B}_m$ and its orthogonal supplement

3. **Skill assessment compared to the Gaussian model:**
   a) 15-20% better ***in twin-data experiments***
   b) 3-7% better ***in real-data experiments***
   [consistent with improvement provided by atmospheric 3dVar hybrids]